Paying an Owed Increase in the Source Pool via a Fixed Payment Ratio

Problem Statement:

We have a *source pool* (an AMM with a constant-product invariant) whose current reserves are (X, Y) so that $X \times Y = K$. The exclusive pool owes an *interest liquidity* (or "Rk increase") to the source pool. Equivalently, we can say the source pool's $\sqrt{k}$-value (which is $\sqrt{K}$) should be increased by some amount interestOwed. Then, the final product of the source pool's reserves must be

$$\text{finalK} \;=\; \bigl(\sqrt{K} + \text{interestOwed}\bigr)^2.$$

We also know the *payment ratio* in which the exclusive pool wants to pay the source pool: that is, if $\mathrm{d}x$ is the amount of token0 paid, and $\mathrm{d}y$ is the amount of token1 paid, we fix the ratio

$$\frac{\mathrm{d}x}{\mathrm{d}y} = r,$$

where r is given (for instance, $r = 2$ means "2 parts token0 for every 1 part token1"). We want to find the $\mathrm{d}x$ and $\mathrm{d}y$ that satisfy both:

1. The final source pool reserves become $\bigl(X + \mathrm{d}x,\; Y + \mathrm{d}y\bigr)$ whose product is finalK.
2. $\mathrm{d}x : \mathrm{d}y$ stays at the fixed ratio r.

Notation:

$$\begin{aligned}
&\text{Current source pool: } (X, Y), \quad K = X \cdot Y, \quad \sqrt{K} = \text{current Rk},\\
&\text{interestOwed} = \Delta, \quad \text{so finalK} = (\sqrt{K} + \Delta)^2,\\
&\mathrm{d}x = \text{amount of token0 paid}, \quad \mathrm{d}y = \text{amount of token1 paid},\\
&\frac{\mathrm{d}x}{\mathrm{d}y} = r \implies \mathrm{d}x = r\,\mathrm{d}y.
\end{aligned}$$

Main Condition (Source Pool Final Product):

$$(X + \mathrm{d}x)\,(Y + \mathrm{d}y) \;=\; \text{finalK}.$$

Substituting $\mathrm{d}x = r\,\mathrm{d}y$, we get:

$$\bigl(X + r\,\mathrm{d}y\bigr)\,\bigl(Y + \mathrm{d}y\bigr) \;=\; \text{finalK}.$$

Expand:

$$(X + r\,\mathrm{d}y)\,(Y + \mathrm{d}y) \;=\; X\,Y \;+\; X\,\mathrm{d}y \;+\; r\,\mathrm{d}y\,Y \;+\; r\,(\mathrm{d}y)^2.$$

We know $X\,Y = K$. So:

$$K + \bigl[X + r\,Y\bigr]\mathrm{d}y + r\,(\mathrm{d}y)^2 = \text{finalK}.$$

Bring all terms to one side:

$$r\,(\mathrm{d}y)^2 + \bigl[X + r\,Y\bigr]\mathrm{d}y + \bigl[K - \mathrm{finalK}\bigr] = 0.$$

Quadratic in $\mathrm{d}y$**:**
Define:

$$\begin{aligned} a &= r, \\ b &= X + r\,Y, \\ c &= K - \mathrm{finalK} = K - (\sqrt{K} + \Delta)^2. \end{aligned}$$

Thus, we solve:

$$r\,(\mathrm{d}y)^2 \;+\; (X + r\,Y)\,\mathrm{d}y \;+\; \bigl[K - (\sqrt{K} + \Delta)^2\bigr] = 0.$$

The standard formula:

$$\mathrm{d}y \;=\; \frac{-\,b \;\pm\; \sqrt{b^2 \;-\; 4\,a\,c}}{2\,a}.$$

Once we find a real root for $\mathrm{d}y \geq 0$, we get

$$\mathrm{d}x = r\,\mathrm{d}y.$$

And then we check if that root is feasible (e.g. does it produce a nonnegative final reserve $(X + \mathrm{d}x) > 0$ and $(Y + \mathrm{d}y) > 0$?).

Numerical Example

Given:

$$X = 100, \quad Y = 200, \quad \sqrt{K} = \sqrt{100 \times 200} = \sqrt{20{,}000} \approx 141.4214,$$

$\mathrm{interestOwed} = \Delta = 30,\quad$ so $\mathrm{finalK} = (\sqrt{K}+\Delta)^2 \approx (141.4214+30)^2 = 171.4214^2 \approx 29{,}392.$ (approx.)

We choose a ratio

$$r = \frac{\mathrm{d}x}{\mathrm{d}y} = 2.0 \quad \text{(meaning 2 parts token0 for each 1 part token1).}$$

Hence the final source pool must be $(X + \mathrm{d}x,\ Y + \mathrm{d}y)$ with product $\approx 29{,}392$.
Form the Quadratic:

$$a = r = 2.0, \quad b = X + r\,Y = 100 + 2\times 200 = 100 + 400 = 500, \quad c = K - \mathrm{finalK} = (100\times 200) - 29{,}392 = 20{,}0$$

So the equation in $\mathrm{d}y$ is

$$2\,(\mathrm{d}y)^2 \;+\; 500\,(\mathrm{d}y) \;-\; 9{,}392 = 0.$$

Thus:

$$\mathrm{d}y = \frac{-500 \;\pm\; \sqrt{(500)^2 - 4\,(2)\,(-9{,}392)}}{2 \cdot 2}.$$

Compute step by step:

$$(500)^2 = 250{,}000,$$

$$4 \cdot 2 \cdot 9{,}392 = 8 \times 9{,}392 = 75{,}136.$$

So the discriminant:

$$\Delta = 250{,}000 - \bigl(-75{,}136\bigr) = 250{,}000 + 75{,}136 = 325{,}136.$$

Hence $\sqrt{\Delta} \approx \sqrt{325{,}136} \approx 570.45$.

$$\mathrm{d}y = \frac{-500 \;\pm\; 570.45}{4}.$$

Two roots:

- $\mathrm{d}y_1 = \frac{-500+570.45}{4} = \frac{70.45}{4} = 17.6125,$
- $\mathrm{d}y_2 = \frac{-500-570.45}{4} = \frac{-1070.45}{4} = -267.6125,$

where the second root is negative and thus meaningless for paying "$\mathrm{d}y \geq 0$." So we discard $\mathrm{d}y_2$.
Choose $\mathrm{d}y = 17.6125$**.** Then

$$\mathrm{d}x = r\,\mathrm{d}y = 2.0 \times 17.6125 = 35.225.$$

Check Final Reserves:

$$X + \mathrm{d}x = 100 + 35.225 = 135.225,$$

$$Y + \mathrm{d}y = 200 + 17.6125 = 217.6125.$$

Product:

$$135.225 \times 217.6125 \;\approx\; 29{,}392,$$

which matches our desired finalK (within rounding).

Interpretation

We found:

$$\mathrm{d}x \approx 35.225, \quad \mathrm{d}y \approx 17.6125,$$

paying them in the ratio 2.0 : 1.0. This increases the source pool's product from $100 \times 200 = 20{,}000$ up to $\approx 29{,}392$, which is the required $\bigl(\sqrt{20{,}000} + 30\bigr)^2$.